Business PreCalculus MATH 1643 Section 004, Spring 2014
Lesson 20: Polynomial Functions

Definition 1. **Quadratic Function:** *A polynomial function of* ***degree*** n *is a function of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_2x^2 + a_1x + a_0,$$

where n *is a* ***nonnegative*** *integer and the* ***coefficients*** a_n, a_{n-1}, $a_{n-2}, \cdots, a_{n-3}$, a_2, a_1, a_0 *are real numbers with* $a_n \neq 0$.

Definition 2. **Leading Term:** *The term* $\underline{a_n x^n}$ *is called the* ***leading term****.*

Definition 3. **Leading Coefficient:** *The number* $\underline{a_n}$ *is called the* ***leading coefficient****.*

Definition 4. **Constant Term:** *The number* $\underline{a_0}$ *is called the* ***constant term****.*

Remark 1. *Note that the* ***domain*** *of a polynomial function is* ***all real numbers*** *or* $(-\infty, \infty)$.

Example 1. *The function* $f(x) = 2x^3 - 5x^4 + 7$ *is a polynomial function of degree* 4*, leading term* $-5x^4$*, leading coefficient* -5*, and constant term* 7.

Definition 5. **Power Function:** *A polynomial function of the form*

$$f(x) = ax^n,$$

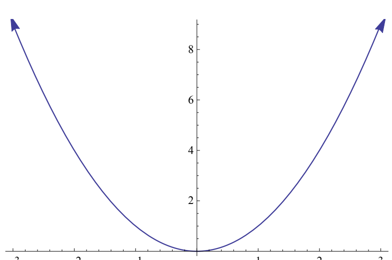
is called a ***power function*** *of degree* n *where* a *is a* ***nonzero*** *real number and* n *is a* ***positive*** *integer.*

Example 2. *The function* $f(x) = 2x^3$ *is a power function of degree* 3.

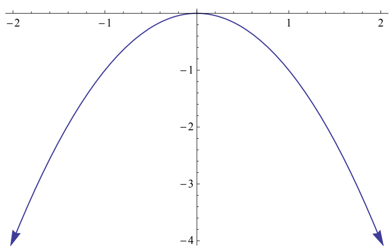
Definition 6. **End Behavior for the Graph of $f(x) = ax^n$:** *The behavior of a function* $y = f(x)$ *as* $x \to \infty$ *or* $x \to -\infty$ *is called the* ***end behavior*** *of the function.*

First*, we will discuss the behavior of* $f(x) = ax^n$ *if* n ***is even****. Since* $\underline{n \textbf{ is even}}$ *then the end behavior of* $f(x) = ax^n$ *is* ***similar*** *to either* $y = x^2$ *or* $y = -x^2$ *depending whether* $a > 0$ *or* $a < 0$.

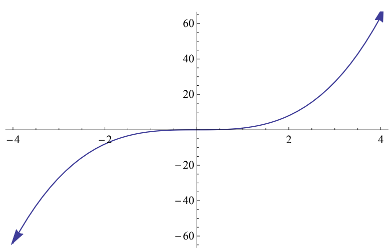
If $a > 0$***, then the graph of*** $f(x) = ax^n$ ***rises to the left and right*** *similar to* $y = x^2$*. We write* $y = f(x) \to \infty$ *as* $x \to \infty$ *and* $y = f(x) \to \infty$ *as* $x \to -\infty$*. Pictorially,*



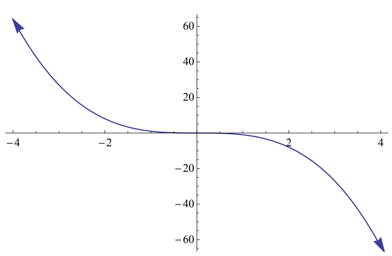
If $a < 0$***, then the graph of*** $f(x) = ax^n$ ***falls to the left and right*** *similar to* $y = -x^2$*. We write* $y = f(x) \to -\infty$ *as* $x \to \infty$ *and* $y = f(x) \to -\infty$ *as* $x \to -\infty$*. Graphically,*



Second*, if* <u>n ***is odd***</u> *then the end behavior of* $f(x) = ax^n$ *is* ***similar*** *to either* $y = x^3$ *or* $y = -x^3$ *depending whether* $a > 0$ *or* $a < 0$*. For* $a > 0$***, the graph of*** $f(x) = ax^n$ ***rises to the right and falls to the left*** *similar to* $y = x^3$*. We write* $y = f(x) \to \infty$ *as* $x \to \infty$ *and* $y = f(x) \to -\infty$ *as* $x \to -\infty$*. Graphically,*



If $a < 0$***, the graph of*** $f(x) = ax^n$ ***rises to the left and falls to the right*** *similar to* $y = -x^3$*. We write* $y = f(x) \to -\infty$ *as* $x \to \infty$ *and* $y = f(x) \to \infty$ *as* $x \to -\infty$*. Pictorially,*



Example 3. *Given the function* $f(x) = 2x^5$*. Since* $n = 3$ *is odd and* $a = 2 > 0$*, then* $f(x)$ *behaves similar to* x^3*. This means that* $f(x) = 2x^5 \to \infty$ *as* $x \to \infty$ *and* $f(x) = 2x^5 \to -\infty$ *as* $x \to -\infty$*.*

Definition 7. <u>**End Behavior for the Graph of Polynomial Function:**</u> *Let* $f(x) = 2x^3 + 5x^2 - 7x + 11$ *be a polynomial function of degree 3. Then the end behavior of* $f(x)$ *is similar to the end behavior of its* ***leading term*** *because:*

$$\begin{aligned} f(x) &= 2x^3 + 5x^2 - 7x + 11 \\ &= x^3\left(2 + \frac{5x^2}{x^3} - \frac{7x}{x^3} + \frac{11}{x^3}\right) \\ &= x^3\left(2 + \frac{5}{x} - \frac{7}{x^2} + \frac{11}{x^3}\right) \end{aligned}$$

$$\approx x^3(2+0-0+0) \qquad \textit{When } x \textit{ is very large}$$
$$= 2x^3.$$

The above definition leads to the following:

Definition 8. **The Leading Term Test:** *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function. Its leading term is $a_n x^n$. The behavior of the graph of $f(x)$ as $x \to \infty$ or $x \to -\infty$ is similar to the behavior of $a_n x^n$.*

Definition 9. **Zeros of Polynomial Functions:** *If f is a polynomial function and c is a real number, then the following statements are equivalent:*

1. *c is a zero of f.*
2. *c is a solution of the equation $f(x) = 0$.*
3. *c is an x-intercept of the graph of $f(x)$.*

Example 4. *Find the zeros of the polynomial function $f(x) = x^3 + 2x^2 - x - 2$.*

Solution: *To find the zeros of $f(x)$, we solve the equation $f(x) = 0$.*

$$\begin{aligned} f(x) &= 0 \\ x^3 + 2x^2 - x - 2 &= 0 \\ x^2(x+2) - 1(x+2) &= 0 \\ (x+2)(x^2-1) &= 0 \\ (x+2)(x+1)(x-1) &= 0. \end{aligned}$$

Then $x = -2$, $x = -1$, or $x = 1$ are the zeros of the function $f(x) = x^3 + 2x^2 - x - 2$.

Remark 2. *A polynomial function of degree n has* ***at most*** *n real zeros.*

Definition 10. **Multiplicity of a Zero:** *If c is a zero of a polynomial function $f(x)$ and the corresponding factor $x - c$ occurs exactly m times when $f(x)$ factored completely, then c is called a* ***zero of multiplicity*** *m.*

Example 5. *Find the zeros of the polynomial function $f(x) = x^2(x+1)(x-2)$ and give the multiplicity of each zero.*

Solution: *To find the zeros of $f(x)$, we solve the equation $f(x) = 0$.*

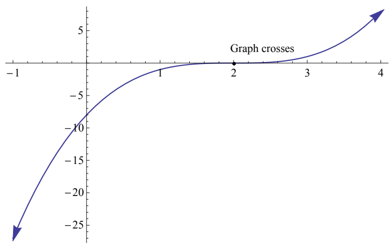
$$\begin{aligned} f(x) &= 0 \\ x^2(x+1)(x-2) &= 0 \end{aligned}$$

Then $x = 0$ is a zero of $f(x)$ with multiplicity 2, $x = -1$ is a zero of $f(x)$ with multiplicity 1 and $x = 2$ is a zero of $f(x)$ with multiplicity 1.

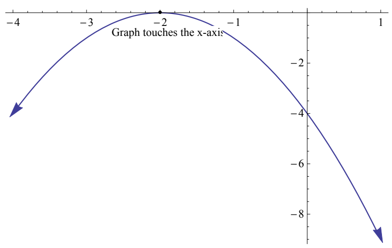
Note *that the degree of this polynomial function is 4.*

Remark 3. *The multiplicity, m, of a zero, c, helps sketch the graph of polynomial functions. We distinguish between the two cases m is even or m is odd.*

1. *If m is odd, the graph of $f(x)$ crosses the x-axis at $x = c$.*



2. *If m is even, the graph of $f(x)$ touches but does not cross the x-axis at $x = c$.*



Definition 11. **<u>Intermediate Value Theorem:</u>** *Let a and b be two numbers with $a < b$. If $f(x)$ is a polynomial function such that $f(a)$ and $f(b)$ have* ***opposite signs****, then there is* ***at least one number*** *c with $\underline{a < c < b}$, for which $\underline{f(c) = 0}$. This means that there is a zero for the function between a and b.*

Example 6. *Show that the function $f(x) = -2x^3 + 4x + 5$ has a real zero between 1 and 2.*

<u>Solution:</u> *To prove the existence of the zero, we need to find $f(1)$ and $f(2)$ and compare their signs.*

$$\begin{aligned} f(1) &= -2(1)^3 + 4(1) + 5 \\ &= 7. \end{aligned}$$

$$\begin{aligned} f(2) &= -2(2)^3 + 4(2) + 5 \\ &= -3. \end{aligned}$$

Since $f(1)$ and $f(2)$ have ***opposite*** *signs, then $f(x) = -2x^3 + 4x + 5$ has a real zero between* 1 *and* 2.